C^1- ACTIONS OF BAUMSLAG-SOLITAR GROUPS ON S^1.

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Abstract. Let $BS(1,n) =< a, b \mid aba^{-1} = b^n >$ be the solvable Baumslag-Solitar group, where $n \geq 2$. It is known that $BS(1,n)$ is isomorphic to the group generated by the two affine maps of the line : $f_0(x) = x+1$ and $h_0(x) = nx$. The action on $S^1 = \mathbb{R} \cup \infty$ generated by these two affine maps f_0 and h_0 is called the standard affine one. We prove that any representation of $BS(1,n)$ into $Diff^1(S^1)$ is (up to a finite index subgroup) semiconjugated to the standard affine action.

1. Introduction.

This paper is about the dynamics of the actions of the solvable Baumslag-Solitar group $-BS(1,n) =< a, b \mid aba^{-1} = b^n >$, where $n \geq 2$– on the circle S^1.

It is known that $BS(1,n)$ has many actions on S^1. The standard action on $S^1 = \mathbb{R} \cup \infty$ is the action generated by the two affine maps $f_0(x) = x + 1$ and $h_0(x) = nx$ (where $f_0 \equiv b$ and $h_0 \equiv a$).

Many people has studied actions of solvable groups on one-manifolds, for example Plante [9], Ghys [4], Navas [6], Farb and Franks[3], Moriyama[5] and Revelo and Silva[10].

There exist some results concerning to $BS(1,n)$-action on S^1. Many of them are scattered in different articles and some of them are not so easy to find, so our aim is to present the state of the art for the case of the action of $BS(1,n)$-group on S^1, in the case C^1.

It is known that any C^2 $BS(1,n)$-action on S^1 admits a finite orbit. This fact was proven by Burslem-Wilkinson in [1]. In fact they gave a classification (up to conjugacy) of representations $\rho : BS(1,n) \to Diff^r(S^1)$ with $r \geq 2$ or $r = \omega$.

These results are proved by using a dynamical approach. The dynamics of C^2 $BS(1,n)$-actions on S^1 is now well understood, due to Navas work on solvable group of circle diffeomorphisms (see [6]). We will prove the same statement of [1] but in the case that the action is C^1, that is, any C^1 $BS(1,n)$-action on S^1 admits a finite orbit.

Our aim is giving a classification of faithful actions of $BS(1,n)$ on S^1, in the case C^1.

So, our main result is the following:

Theorem. *Any $C^1 - BS(1,n)$ action on S^1 preserving orientation, $< f, h >$, is (up to a finite index subgroup $< f^{n-1}, h^m >$) semiconjugated to the standard affine action.*

The proof uses in a crucial way (a slightly extended version of) an argument due to Cantwell and Conlon ([2]), and its use was suggested to us by A. Navas. It should be pointed out that, using this argument, Navas has recently obtained a counter-example to

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the converse of Thurston Stability Theorem (see [8]), and it is very likely that Cantwell-Conlon argument will still reveal very fruitful for obtaining new obstructions to C^1 actions on 1-dimensional manifolds

2. Existence of a global finite orbit.

As before, let $< f, h >$ be a faithful action of $BS(1, n)$ on S^1, where $h \circ f \circ h^{-1} = f^n$. The aim of this section is proving the existence of a common periodic point of f and h. We begin by proving that the rotation number of f is rational.

Proposition 2.1. *Let $f, h : S^1 \to S^1$ be homeomorphisms preserving orientation verifying $h \circ f \circ h^{-1} = f^n$. Then there exists $l \in \mathbb{N}$ such that the rotation number of f, $\rho(f) = \frac{l}{n-1}$.*

Proof:
Since $h \circ f \circ h^{-1} = f^n$, we have that there exists $l \in \mathbb{Z}$ such that

$$n\rho(f) = \rho(f) + l,$$

then $\rho(f) = l/(n-1)$. ■

We will prove that not only the rotation number of f is rational but also the rotation number of h it is.

Proposition 2.2. *Let $f, h : S^1 \to S^1$ be C^1- diffeomorphisms preserving orientation verifying $h \circ f \circ h^{-1} = f^n$. If the $BS(1, n)-$ action on S^1, $< f, h >$, is faithful then $\rho(h)$, the rotation number of h, is rational.*

We suppose that $\rho(h)$ is irrational, then we have two cases:

(1) h is conjugated to an irrational rotation. Note that the periodic points of f are preserved by h: let q such that $f^k(q) = q$, then $f^{nk}(h(q)) = h(f^k(q)) = h(q)$. So $h(q)$ is a periodic point for f.
It follows that the periodic points of f are dense in S^1. This implies that there exists m such that $f^m = Id$ contradicting that the action is faithful.

(2) The minimal set of h is a Cantor set, K. Then there exists an arc $J \subset K^c \subset S^1$ fixed by f (its end points are fixed) where $f|_J \neq Id$ and length of J is very small. Let $\epsilon > 0$ verifying $(1-\epsilon)^2 > \frac{3}{4}$, we choose J sufficiently small in order to $length(h^{-s}(J))$ (it is possible since J is an h-wandering interval) is small enough to guarantee $f' \geq 1 - \epsilon$ for any $x \in J$ and $\frac{h'(x)}{h'(y)} \geq 1 - \epsilon$ for any $x, y \in J$ and also $f' \geq 1 - \epsilon$ for any $x \in h^{-s}(J)$ and $\frac{h'(x)}{h'(y)} \geq 1 - \epsilon$ for any $x, y \in h^{-s}(J)$, for any $s > 0$.
Let $x \in J$ and $I \subset J$ be the open arc between x and $f(x)$. It is easy to see that $\cup_{k\in\mathbb{Z}} f^k(I) \subset J$ and $f^k(I) \cap f^j(I) = \emptyset$ if $k \neq j$. Let us define the map $\Psi(\alpha_0, \alpha_1, \alpha_2, ..., \alpha_m)$ with $\alpha_i \in \{0, 1\}$ as the map

$$(h^m \circ f^{\alpha_m} \circ h^{-m}) \circ ... \circ (h^2 \circ f^{\alpha_2} \circ h^{-2}) \circ (h \circ f^{\alpha_1} \circ h^{-1}) \circ f^{\alpha_0}.$$

Since $h^i \circ f^{\alpha_i} \circ h^{-i} = f^{n^i}$ if $\alpha_i = 1$ and $h^i \circ f^{\alpha_i} \circ h^{-i} = Id$, otherwise; it follows that $\Psi(\alpha_0, \alpha_1, \alpha_2, ..., \alpha_m)(I) = f^\beta(I)$ where $\beta = \sum_{i|\alpha_i=1} n^i$.

Therefore, for $(\alpha_0, \alpha_1, \alpha_2, ..., \alpha_m) \neq (\alpha'_0, \alpha'_1, \alpha'_2, ..., \alpha'_m)$ it holds that $\Psi(\alpha_0, \alpha_1, \alpha_2, ..., \alpha_m)(I) \cap \Psi(\alpha'_0, \alpha'_1, \alpha'_2, ..., \alpha'_m)(I)$ is empty. Hence,

$$\mathcal{I} = \bigcup_{\{0,1\}^{m+1}} \Psi(\alpha_0, \alpha_1, \alpha_2, ..., \alpha_m)(I)$$

is the union of 2^{m+1} disjoint arcs included in J.

We claim that the length of any arc $\Psi(\alpha_0, \alpha_1, \alpha_2, ..., \alpha_m)(I)$ may be lower bounded by $|I|(\frac{3}{4})^{m+1}$:

Notice that

$$\Psi(\alpha_0, \alpha_1, \alpha_2, ..., \alpha_m) = h^k \circ f \circ h^{-l_1} \circ f \circ h^{-l_2} \circ f \circ ... \circ f \circ h^{-l_r}$$

where $l_1 + l_2 + ... + l_r = k$, $l_i > 0$ for $i = 1, ..., r-1$ and $l_r \geq 0$. Hence,

$$\Psi'(\alpha_0, \alpha_1, \alpha_2, ..., \alpha_m)(u) = \prod_{i=1}^{r} f'(x_i) \prod_{j=1}^{k} \frac{h'(y_j)}{h'(w_j)},$$

where x_i, y_j and w_j are well defined points, $x_i \in \cup_{s\in\mathbb{N}} h^{-s}(J)$ and $y_j, w_j \in h^{-j}(J)$.

Therefore, there exist points $\widehat{x_i}$ for $i = 1, .., r$, $\widehat{y_j}$ and $\widehat{w_j}$ for $j = 1, .., k$ such that the length of $\Psi(\alpha_0, \alpha_1, \alpha_2, ..., \alpha_m)(I)$,

$$|\Psi(\alpha_0, \alpha_1, \alpha_2, ..., \alpha_m)(I)| = |I| \prod_{i=1}^{r} |f'(\widehat{x_i})| \prod_{j=1}^{k} \frac{|h'(\widehat{y_j})|}{|h'(\widehat{w_j})|} \geq |I|(1-\epsilon)^{r+k}.$$

Since $r \leq k \leq m+1$ it follows that

$$|\Psi(\alpha_0, \alpha_1, \alpha_2, ..., \alpha_m)(I)| \geq |I|(1-\epsilon)^{2m+2} \geq |I|(\frac{3}{4})^{m+1}.$$

Then, the length of $\mathcal{I} \geq 2^{m+1}|I|(\frac{3}{4})^{m+1}$ which tends to ∞ when $m \to \infty$. This is a contradiction with $\mathcal{I} \subset J$.

Therefore, the minimal set of h is not a Cantor set.

Hence, we have proved that $\rho(h)$ is rational.$\blacksquare$

Proposition 2.3. *There exists $m \in \mathbb{N}$ such that f^{n-1} and h^m have a common fixed point*

Proof: Let m such that $\rho(h^m) = 0$. Let q be a fixed point for f^{n-1} (we have already seen that there exists a fixed point for f^{n-1}), then $\{h^{lm}(q)\}$ is included in the set of fixed points of f^{n-1} since h preserves the fixed points of f^{n-1}.

Let u be an accumulation point of $\{h^{lm}(q)\}$. It follows from continuity of f that u is a fixed point for f^{n-1} and since $\Omega(h^m)$ is included in the set of fixed points of h^m then u is also a fixed point for h^m.$\blacksquare$

Since the C^1 diffeomorphisms f^{n-1} and h^m have a common fixed point, we can study $< f^{n-1}, h^m >$, a C^1-action of $BS(1,(n-1)n^m)$ on the interval $[0,1]$ instead of S^1.

3. Semiconjugation to the standard affine action.

Recall that the standard affine action on $S^1 = \mathbb{R} \cup \infty$ is the action generated by the two affine maps $f_0(x) = x + 1$ and $h_0(x) = nx$.

Following results or ideas of Cantwell-Conlon, Navas and Rivas, from now on we will prove that any faithful $C^1 - BS(1,n)$ action on S^1 is semiconjugated to the standard affine action.

Due to a classical result that appears, for example in "Groups acting on the circle" by E. Ghys (see [4]), it is known that for a countable infinite group G, G is left orderable if and only if G acts faithfully on the real line by orientation preserving homeomorphisms. Let us note that for proving that G acts faithfully on the real line E. Ghys constructed the **dynamical realization of a left ordering**. For this construction he fixed an enumeration $\{g_i\}$ of G. He defined an order preserving map $t : G \to \mathbb{R}$, in such a way that $g(t(g_i)) = t(gg_i)$. This action was extended to the closure of $t(G)$, and later to the whole real line. As Rivas noted (see remark 4.4 of [11]), and it was proved by Navas (see [7]) that the dynamical realization associated to differents enumerations of G (but the same ordering) are topological conjugated.

It was proven by Rivas (see [11])) that the set of left orderings of $BS(1,n)$ is made up of four Conradian orderings and an uncountable set of non Conradian left orderings. Each one of these infinetely many non Conradian orderings can be realized as an induced ordering that comes from an affine action on $BS(1,n)$. Moreover, in the proof of this result it was shown that the dynamical realization of any non Conradian ordering is semiconjugated to the standard affine action. (In fact, Rivas proved this result for $BS(1,2)$ but the proof for $BS(1,n)$ is the same).

As Ghys and Rivas results are "topological", they hold in an interval instead of the real line.

We will call an "exotic" action to the $BS(1,n)$ one that is induced by a Conradian ordering.

Let $\ll f \gg$ be the largest abelian subgroup containing f. For any "exotic" action in the interval $[0,1]$ (that is induced by one of the four Conradian orderings), $\ll f \gg$ is a convex subgroup (in the sense of ordening, see the proof of Proposition 4.1 of [11]). Then, for a point x_0 in $[0,1]$, the sequence $\{f^n(x_0)\}$ is lower and upper bounded, then there exist $\lim_{n\to\infty} f^n(x_0) = d < 1$ and $\lim_{n\to-\infty} f^n(x_0) = c > 0$ two f- fixed points verifying that there is no fixed point for f in (c,d). Moreover, $h(c,d)$ is disjoint of (c,d).

It was proven by Cantwell and Conlon ([2]) that any "exotic" BS- action on the interval I can not be C^1. In fact, the proof of item (2) of Proposition 2.2 follows Cantwell-Conlon's proof.

It follows that any C^1 BS- action on an interval I is semiconjugated to the standard affine action. The same holds for C^1 BS- action on S^1.

So, we have proved

Theorem. *Any $C^1 - BS(1,n)$ action on S^1 is (up to a finite index subgroup $< f^{n-1}, h^m >$) semiconjugated to the standard affine action.*

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